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An asymptotic analysis of laminar film boiling on vertical plates including variable property effects

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Abstract—Laminar film boiling is studied as an example to demonstrate the advantages of the asymptotic approach to rather complex two-phase flow and heat transfer problems. Introducing two perturbation parameters for superheating and subcooling, respectively, a regular perturbation solution with only two solution parameters is derived. The effects of variable properties are included asymptotically. Non-asymptotic solutions are far less general since even for constant properties it is a six-parameter problem which allows for specific solutions only.

1. INTRODUCTION

ANALYTICAL studies of laminar film boiling date back to an early study by Bromley [1] in 1950. Assuming a linear temperature distribution in the vapour film this study was based on a modification of the Nusselt water-film theory. About 10 years later considerable improvement was achieved by introducing the concept of a two-phase boundary layer, see, e.g. studies by Sparrow and Cess [2], Nishikawa and Ito [3] and Frederking and Hopenfeld [4]. Though nearly all studies point out that the influence of variable properties might be very important for that kind of flow only a few of them account for the temperature dependence of the physical properties involved. McFadden and Grosh [5] account for variable density and specific heat for a flow situation near the critical pressure. A study assuming all properties to vary with temperature (expressed by a power series representation) has been published by Marschall and Moresco [6].

In the present study laminar stable two-phase boundary layer flow on a vertical plate is investigated. The surface of the plate is heated above the saturation temperature of the surrounding liquid. In general the liquid is subcooled including the special limit of zero subcooling or saturation.

In reality this flow situation holds close to the leading edge of the plate. In regions further downstream where deformations of the interface by waves and instabilities arise, the present analysis may provide the basic flow for stability considerations.

The objective of the present study is a more systematic approach to the problem. The application of a regular perturbation technique provides more general results as well as better physical insight into the problem. Based on this asymptotic approach the influence of the various temperature-dependent properties is revealed clearly. In addition to this, well-known variable property concepts like the 'reference temperature'—and 'property ratio'—method can be adopted to this flow situation.

Compared to the treatment of a one-phase natural convection boundary layer, see, e.g. ref. [7], two additional complications arise. The interface location and conditions must be expressed asymptotically; and—as a consequence of this—a series expansion of the vapour boundary layer is required in addition to the expansion accounting for variable property effects.

2. ANALYSIS

The two-phase boundary layer flow, according to Fig. 1, is described by the following set of conservation equations. All starred quantities are dimensional, quantities in the liquid phase are marked with a circumflex '^':

continuity

$$\frac{\partial}{\partial x^*}(\rho^* u^*) + \frac{\partial}{\partial y^*}(\rho^* v^*) = 0 \tag{1a}$$

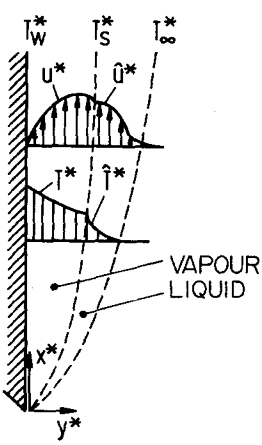

FIG. 1. Two-phase boundary layer.

NOMENCLATURE

C density ratio, $\hat{\rho}^*_\infty/\rho^*_s$
c_p specific heat at constant pressure
f dimensionless streamfunction
g acceleration of gravity
Gr Grashof number
h latent heat
H latent heat, Table 2
j reference temperature factor
K_α property of the fluid, equation (40)
$K_A, \hat{K}_A$ constants, equations (29)
$\dot{m}$ mass flux
n_α property ratio exponent
Nu Nusselt number
Pr Prandtl number
q heat flux
R Reynolds number
T temperature
u, v velocity components
U reference velocity
x, y coordinates.

Greek symbols
α physical property
ε overheating parameter, equation (12a)
$\hat{\varepsilon}$ subcooling parameter, equation (12b)
η similarity variable
θ dimensionless temperature
λ thermal conductivity
μ viscosity
ρ density
τ shear stress
ψ streamfunction.

Subscripts
α associated with the property $\alpha(\rho, \eta, \ldots)$
cp constant properties
IF interface
s saturation
w wall
∞ infinity.

$$\frac{\partial}{\partial x^*}(\hat{\rho}^*\hat{u}^*)+\frac{\partial}{\partial y^*}(\hat{\rho}^*\hat{v}^*)=0; \tag{1b}$$

x-momentum

$$\rho^*\left(u^*\frac{\partial u^*}{\partial x^*}+v^*\frac{\partial u^*}{\partial y^*}\right)=\frac{\partial}{\partial y^*}\left(\mu^*\frac{\partial u^*}{\partial y^*}\right)+g^*(\hat{\rho}^*_\infty-\rho^*) \tag{2a}$$

$$\hat{\rho}^*\left(\hat{u}^*\frac{\partial \hat{u}^*}{\partial x^*}+\hat{v}^*\frac{\partial \hat{u}^*}{\partial y^*}\right)=\frac{\partial}{\partial y^*}\left(\hat{\mu}^*\frac{\partial \hat{u}^*}{\partial y^*}\right)+g^*(\hat{\rho}^*_\infty-\hat{\rho}^*); \tag{2b}$$

thermal energy (viscous heating and axial conduction neglected)

$$\rho^* c_p^*\left(u^*\frac{\partial T^*}{\partial x^*}+v^*\frac{\partial T^*}{\partial y^*}\right)=\frac{\partial}{\partial y^*}\left(\lambda^*\frac{\partial T^*}{\partial y^*}\right) \tag{3a}$$

$$\hat{\rho}^*\hat{c}_p^*\left(\hat{u}^*\frac{\partial \hat{T}^*}{\partial x^*}+\hat{v}^*\frac{\partial \hat{T}^*}{\partial y^*}\right)=\frac{\partial}{\partial y^*}\left(\hat{\lambda}^*\frac{\partial \hat{T}^*}{\partial y^*}\right). \tag{3b}$$

The associated boundary conditions are

$$y^*=0:\quad u^*=v^*=T^*-T^*_w=0 \tag{4}$$

$$y^*\to\infty:\quad \hat{u}^*=\hat{T}^*-\hat{T}^*_\infty=0. \tag{5}$$

In addition to the five boundary conditions another six conditions hold at the (unknown) interface position y^*_{IF}, for details see, e.g. ref. [6]

$$u^*=\hat{u}^*\quad \text{(no-slip velocity)} \tag{6}$$

$$\mu^*\frac{\partial u^*}{\partial y^*}=\hat{\mu}^*\frac{\partial \hat{u}^*}{\partial y^*}\quad \text{(balance of forces)} \tag{7}$$

$$\rho^*\left(u\frac{\partial y^*_{IF}}{\partial x^*}-v^*\right)=\hat{\rho}^*\left(\hat{u}^*\frac{\partial y^*_{IF}}{\partial x^*}-\hat{v}^*\right)\quad \text{(mass conservation)} \tag{8}$$

$$T^*=T^*_s \tag{9}$$

(no temperature jump)

$$\hat{T}^*=T^*_s \tag{10}$$

$$\lambda^*\frac{\partial T^*}{\partial y^*}=\hat{\lambda}^*\frac{\partial \hat{T}^*}{\partial y^*}-\frac{\dot{m}^*}{A^*}h^*\ \text{(energy conservation)}. \tag{11}$$

By equations (4)–(11) eleven conditions are provided, ten boundary conditions for equations (1)–(3) and one condition to determine y^*_{IF}. The interface is assumed to be at the saturation state determined by the saturation temperature T^*_s at the total pressure p^*.

A regular two-parameter perturbation analysis is now applied to the system of equations (1)–(11). Since the two-phase flow under consideration is induced by overheating the fluid the rate of overheating is chosen as one perturbation parameter

$$\varepsilon=\frac{T^*_w-T^*_s}{T^*_s}\quad \text{(overheating parameter)}. \tag{12a}$$

In addition there is a modification of the two-phase flow by the degree of subcooling, so that the second perturbation parameter is defined as

$$\hat{\varepsilon}=\frac{T^*_\infty-T^*_s}{T^*_s}\quad \text{(subcooling parameter)}. \tag{12b}$$

The following asymptotic theory holds for $\varepsilon\to 0$, $\hat{\varepsilon}\to 0$ but it will turn out that reasonable approximations for finite ε and $\hat{\varepsilon}$ will be achieved even if only the first (linear) term of the asymptotic expansion is considered. A five-step procedure will be applied in Sections 2.1–2.5. For a general description of the five steps see, e.g. ref. [8] or ref. [9].

Table 1. Non-dimensional quantities

x	$y, \hat{y}$	$u, \hat{u}$	$v, \hat{v}$	$\psi, \hat{\psi}$	$\theta, \hat{\theta}$	$\alpha, \hat{\alpha}$
$\frac{x^*}{L^*}$	$\frac{y^*}{L^*}R^{1/2}$	$\frac{u^*}{U^*}$	$\frac{v^*}{U^*}R^{1/2}$	$\frac{\psi^* R^{1/2}}{\rho_s^* U^* L^*}$	$\frac{T^*-T_s^*}{T_w^*-T_s^*}$	$\frac{\alpha}{\alpha_s^*}$
	$\frac{y^*-y^*_{\text{IFO}}}{L^*}\hat{R}^{1/2}$	$\frac{\hat{u}^*}{\hat{U}^*}$	$\frac{\hat{v}^*}{\hat{U}^*}\hat{R}^{1/2}$	$\frac{\hat{\psi}^* \hat{R}^{1/2}}{\hat{\rho}_s^* \hat{U}^* L^*}$	$\frac{\hat{T}^*-T_s^*}{T_\infty^*-T_s^*}$	$\frac{\hat{\alpha}^*}{\hat{\alpha}_s^*}$

$\alpha = \rho, \mu, \ldots; \hat{\alpha} = \hat{\rho}, \hat{\mu}, \ldots$
$R = (\rho_s^* U^* L^*/\mu_s^*); \hat{R} = (\hat{\rho}_s^* \hat{U}^* L^*/\hat{\mu}_s^*); Pr = (\mu_s^* c_{ps}^*/\lambda_s^*); \hat{Pr} = (\hat{\mu}_s^* \hat{c}_{ps}^*/\hat{\lambda}_s^*)$.

2.1. *Non-dimensional basic equations, similarity transformation*

Since there is no characteristic geometrical length in the problem under consideration the flow exhibits self-similar behaviour. Equations (1)–(3) therefore can be transformed to a set of ordinary differential equations.

(1) Nondimensionalization, using an arbitrary reference length L^*, reference velocities U^* and $\hat{U}^*$, and T_s^* as reference temperature. All physical properties are nondimensionalized by their values at T_s^*. For details see Table 1.

(2) Applying the following similarity transformations: $(x, y) \to (x_s, \eta)$ and $(x, \hat{y}) \to (x_s, \hat{\eta})$

$$x_s = x \tag{13}$$

$$\eta = 2^{-1/2} x^{-1/4} \varepsilon^{-n} \int_0^y \rho \, \mathrm{d}y \tag{14a}$$

$$\hat{\eta} = 2^{-1/2} x^{-1/4} \int_0^{\hat{y}} \hat{\rho} \, \mathrm{d}\hat{y}. \tag{14b}$$

(3) Introducing the streamfunctions ψ and $\hat{\psi}$ and their self-similar counterpart f and $\hat{f}$

$$\frac{\partial \psi}{\partial y} = \rho u, \quad \frac{\partial \psi}{\partial x} = -\rho v, \quad \psi = 2\sqrt{2} x^{3/4} \varepsilon^n f(\eta) \tag{15a}$$

$$\frac{\partial \hat{\psi}}{\partial y} = \hat{\rho}\hat{u}, \quad \frac{\partial \hat{\psi}}{\partial x} = -\hat{\rho}\hat{v}, \quad \hat{\psi} = 2\sqrt{2} x^{3/4} \hat{f}(\hat{\eta}). \tag{15b}$$

Equations (13)–(15) are those frequently used in two-phase self-similar flows, but with two additional features. In equations (14) a density weighted normal coordinate is introduced for convenience. In equations (14a) and (15a) the vapour quantities are stretched by a power of ε. The reason for this additional transformation arises from the different structure of the buoyancy terms in the liquid and the vapour phases. The exponent n will be evaluated by physical considerations.

With the buoyancy terms named BT and BTL, respectively, equations (1)–(3) now read:

x-momentum

$$[\rho\mu f'']' + \varepsilon^{2n}[3ff'' - 2f'^2 + \text{BT}] = 0 \tag{16a}$$

$$[\hat{\rho}\hat{\mu}\hat{f}'']' + 3\hat{f}\hat{f}'' - 2\hat{f}'^2 + \text{BTL} = 0; \tag{16b}$$

thermal energy

$$[\rho\lambda\theta']' + \varepsilon^{2n} 3 Pr\, c_p f \theta' = 0 \tag{17a}$$

$$[\hat{\rho}\hat{\lambda}\hat{\theta}']' + 3\hat{Pr}\, \hat{c}_p \hat{f}\hat{\theta}' = 0; \tag{17b}$$

with the associated boundary conditions

$$\eta = 0: \quad f = f' = \theta - 1 = 0 \tag{18}$$

$$\hat{\eta} \to \infty: \quad \hat{f}' = \hat{\theta} - 1 = 0 \tag{19}$$

and the conditions at the (unknown) interface position η_{IF} listed at the end of this section.

The buoyancy terms describing the driving mechanism of the flow actually are the key to asymptotic formulation of the whole problem. They read with the driving density ratio $C \equiv \hat{\rho}_\infty^*/\rho_s^*$

$$\text{BT} = \frac{g^* L^*}{U^{*2}}[C\rho^{-1} - 1] = \frac{g^* L^*}{U^{*2}}[(C-1) + O(\varepsilon)] \tag{20a}$$

$$\text{BTL} = \frac{g^* L^*}{\hat{U}^{*2}}[\hat{\rho}_\infty \hat{\rho}^{-1} - 1] = \frac{g^* L^*}{\hat{U}^{*2}}[\hat{\varepsilon}\hat{K}_\rho(1-\hat{\theta}) + O(\hat{\varepsilon}^2)]. \tag{20b}$$

Since one is looking for an asymptotic solution for $\varepsilon \to 0$, $\hat{\varepsilon} \to 0$, ρ and $\hat{\rho}$ in equations (20) have been replaced by their Taylor series expansions at the reference temperature T_s^*

$$\rho = 1 + \varepsilon K_\rho \theta + O(\varepsilon^2) \tag{21a}$$

$$\hat{\rho} = 1 + \hat{\varepsilon}\hat{K}_\rho \hat{\theta} + O(\hat{\varepsilon}^2) \tag{21b}$$

with the following definitions:

$$K_\rho = \left[\frac{T^*}{\rho^*}\frac{\partial \rho^*}{\partial T^*}\right]_s, \quad \hat{K}_\rho = \left[\frac{T^*}{\hat{\rho}^*}\frac{\partial \hat{\rho}^*}{\partial T^*}\right]_s. \tag{22}$$

These values are dimensionless fluid properties like the Prandtl numbers Pr and $\hat{Pr}$.

The constant property solution in the limit $\varepsilon \to 0$ is assumed to be the zero-order or basic solution for the perturbation approach. Compared to the single phase natural convection flow this is a completely different approach. For a single phase flow the zero-order solution is a variable property solution already (known as the Boussinesq approximation, for details see ref. [7]). For two-phase flows a finite buoyancy term is left in the momentum equation of the vapour phase, see equation (2a), even in the limit of constant properties.

There is still the large difference in density between the vapour and liquid phases.

From equation (20a) the reference velocity of the vapour phase is determined. In order to keep the buoyancy term in the zero-order equations, for U^* to hold, see equation (16a)

$$U^*[g^*L^*(C-1)]^{-1/2} = O(\varepsilon^n). \quad (23)$$

At this stage of the investigation it is not obvious what fixes the unknown exponent n. It turns out that this is done by the energy balance at the interface, equation (11). To zero order it reads (0, zero order; $H \equiv h^*/c^*_{ps}T^*_s$)

$$\theta'_0 + 3Pr\, Hf_0\varepsilon^{2n-1} = 0. \quad (24)$$

Since both terms in equation (24) must be kept in the equation (the latent heat produced at the interface is balanced by the heat conduction to the vapour face) one obtains

$$\varepsilon^{2n-1} = 1, \quad n = 1/2. \quad (25)$$

With $n = 1/2$ the reference velocity of the vapour phase is (with a possible order one constant equal to one)

$$U^* = \sqrt{(g^*L^*(C-1)\varepsilon)}. \quad (26)$$

Within the zero-order solution (constant properties) there are no free convection currents in the liquid. But nevertheless there is a liquid boundary layer since the no-slip condition provides a non-zero boundary condition to the otherwise homogeneous equation (2b) (note: $\hat{\rho}^*_\infty - \hat{\rho}^* = 0$ in equation (2b) within zero order). As a consequence of this the reference velocity in the liquid phase must be of the same order of magnitude as U^*. To avoid another $O(1)$ constant, one sets

$$\hat{U}^* = U^*. \quad (27)$$

Equations (16) with BT and BTL according to equations (20)–(26) now read

$$[\rho\mu f'']' + 1 + \varepsilon[3ff'' - 2f'^2 - K_A\theta] + O(\varepsilon^2) = 0 \quad (28a)$$

$$[\hat{\rho}\hat{\mu}\hat{f}'']' + 3\hat{f}\hat{f}'' - 2\hat{f}'^2 + \frac{\hat{\varepsilon}}{\varepsilon}\hat{K}_A(1-\hat{\theta}) + O\left(\frac{\hat{\varepsilon}^2}{\varepsilon}\right) = 0 \quad (28b)$$

with

$$K_A = \frac{CK_\rho}{C-1}, \quad \hat{K}_A = \frac{\hat{K}_\rho}{C-1}. \quad (29)$$

In equation (28b) the buoyancy term of $O(\hat{\varepsilon}/\varepsilon)$ describes the free convection currents in the liquid which are one of several variable property effects. By assuming the constant property case for $\varepsilon \to 0$ to be the zero-order solution the free convection currents as a consequence of this assumption are the only *first-order* effect in equation (28b). A necessary condition imposed by the zero-order assumption is $O(\hat{\varepsilon}) = O(\varepsilon^m)$ with $m > 1$. For $m = 1$ the free convection currents would be a zero-order effect (which is discussed briefly later). Any number $m > 1$ is possible since ε and $\hat{\varepsilon}$ are not related to one another by a physical condition that would allow for one combination only. But nevertheless there is a first choice

$$m = 2, \quad O(\hat{\varepsilon}) = O(\varepsilon^2), \quad \hat{\varepsilon} = P_1\varepsilon^2. \quad (30)$$

Analysing the whole problem incorporating equations (30) allows for a minimum number of expansion terms as will be seen hereafter. Furthermore, the well-established hierarchy of variable property effects is preserved (see, e.g. ref. [7]): the (variable property) buoyancy term in equation (28b) is one order of magnitude larger than the effect of variable viscosity for example. An analysis based on equations (30) seems reasonable for practical purposes since in most applications ε is much larger than $\hat{\varepsilon}$ numerically. For example $T^*_w - T^*_s = 100$ K, $T^*_s - T^*_\infty = 30$ K and $T^*_s = 370$ K results in $\varepsilon = 0.27$, $\hat{\varepsilon} = 0.08$ with $\varepsilon^2 = 0.073$.

The complete set of basic equations needed for a linear asymptotic analysis reads

$$[\rho\mu f'']' + 1 + \varepsilon[3ff'' - 2f'^2 - K_A\theta] + o(\varepsilon) = 0 \quad (31a)$$

$$\hat{f}'' + 3\hat{f}\hat{f}'' - 2\hat{f}'^2 + \frac{\hat{\varepsilon}}{\varepsilon}\hat{K}_A(1-\hat{\theta}) + o(\hat{\varepsilon}/\varepsilon) = 0 \quad (31b)$$

$$[\rho\lambda\theta']' + \varepsilon\, 3Pr\, f\theta' + o(\varepsilon) = 0 \quad (32a)$$

$$\hat{\theta}'' + 3\hat{P}r\, \hat{f}\hat{\theta}' + o(\hat{\varepsilon}/\varepsilon) = 0 \quad (32b)$$

with boundary conditions (18) and (19). The symbol $o(\ldots)$ means asymptotically smaller than the order indicated.

It should be mentioned that a basically different approach is possible by assuming a zero-order solution that incorporates the liquid currents. This results in an analysis assuming $\varepsilon \to 0$, $\hat{\varepsilon} \to 0$, $(\hat{\varepsilon}/\varepsilon) \equiv P_1 = O(1)$. The appropriate reference velocity in the liquid would be $\hat{U}^* = \sqrt{(g^*L^*\hat{K}_\rho\hat{\varepsilon})}$ and all terms of $O(\hat{\varepsilon})$ in equations (31b) and (32b) would turn out to be first-order terms. In terms of this approach the present analysis is that for $P_1 \to 0$ with $P_1 = O(\varepsilon)$.

The interfacial conditions, equations (6)–(11), read (constants listed in Table 2)

$$f' = \hat{f}' \quad (33)$$

$$f'' = \hat{f}''C_1\varepsilon^{1/2} \quad (34)$$

$$C_1\varepsilon^{1/2}f = \hat{f} \quad (35)$$

$$\theta = 0 \quad (36)$$

$$0 = \hat{\theta} \quad (37)$$

Table 2. Constants

$C_1 = \sqrt{\left(\dfrac{\hat{\rho}^*_s\hat{\mu}^*_s}{\rho^*_s\hat{\mu}^*_s}\right)}$	$C_3 = \dfrac{\lambda^*_s}{\hat{\lambda}^*_s}\sqrt{\left(\dfrac{\rho^*_s\mu^*_s}{\hat{\rho}^*_s\hat{\mu}^*_s}\right)}$
$C_2 = \sqrt{\left(\dfrac{\hat{\rho}^*_s\mu^*_s}{\rho^*_s\hat{\mu}^*_s}\right)}$	$H = \dfrac{h^*}{c^*_{ps}T^*_s}$

$$\theta' + 3Pr\, fH = \hat{\theta}' C_3 \varepsilon^{1/2} \frac{\hat{\varepsilon}}{\varepsilon}. \tag{38}$$

2.2. *Perturbation expansions for all dependent variables and the interface location*

It is the aim of this study to find a solution that includes all effects of $O(\varepsilon)$ and $O(\hat{\varepsilon}/\varepsilon)$ in a way that is as general as possible. The first step to accomplish this is a Taylor series expansion of all properties involved. These are $\alpha = \rho\mu$, $\rho\lambda$ (note that c_p is multiplied by ε in equation (17a) and no longer appears in equation (32a)) at the vapour side

$$\alpha = 1 + \varepsilon K_\alpha \theta + O(\varepsilon^2) \tag{39}$$

with

$$K_\alpha = \left[\frac{T^*}{\alpha^*}\frac{\partial \alpha^*}{\partial T^*}\right]_s, \quad \alpha = \rho\mu, \rho\lambda. \tag{40}$$

The influences of all corresponding properties $\hat{\alpha}$ at the liquid side are of $O(\hat{\varepsilon}) = O(\varepsilon^2)$ neglected in this study as higher order terms. Altogether there are three sources of non-zero order terms that should be accounted for in a general perturbation formula:

(a) terms induced by powers of the perturbation parameters in the equations or boundary conditions that hold even for constant property flows (all values of K equal to zero); these are terms of $O(\varepsilon^{1/2})$ and $O(\varepsilon)$;

(b) terms accounting for variable property effects in the vapour layer; these are terms of $O(K_A\varepsilon)$ and $O(K_\alpha\varepsilon)$;

(c) terms accounting for variable property effects in the liquid layer; this is only one term of $O(\hat{K}_A\hat{\varepsilon}/\varepsilon)$.

Accordingly the perturbation expansions for f and $\hat{f}$ read with the constants C_1, C_2 and the values of K arranged properly in order to minimize the parameters of the final equations

$$f = f_0 + \varepsilon^{1/2}C_1 f_1 + \varepsilon[C_1^2 f_2 + C_1C_2 f_3 + f_{41} + Pr\, f_{42}] + \varepsilon\left[K_A f_A + \sum_\alpha K_\alpha f_\alpha\right] + \frac{\hat{\varepsilon}}{\varepsilon}\hat{K}_A f_{\hat{A}} + O(\varepsilon^{3/2}) \tag{41}$$

$$\hat{f} = \hat{f}_0 + \varepsilon^{1/2}C_1 \hat{f}_1 + \varepsilon[C_1^2 \hat{f}_2 + C_1C_2 \hat{f}_3 + \hat{f}_{41} + Pr\, \hat{f}_{42}] + \varepsilon\left[K_A \hat{f}_A + \sum_\alpha K_\alpha \hat{f}_\alpha\right] + \frac{\hat{\varepsilon}}{\varepsilon}\hat{K}_A \hat{f}_{\hat{A}} + O(\varepsilon^{3/2}). \tag{42}$$

Similar expansions hold for the temperatures

$$\theta = \theta_0 + \varepsilon^{1/2}C_1 \theta_1 + \varepsilon[C_1^2 \theta_2 + C_1C_2 \theta_3 + \theta_{41} + Pr\, \theta_{42}] + \varepsilon\left[K_A \theta_A + \sum_\alpha K_\alpha \theta_\alpha\right] + \frac{\hat{\varepsilon}}{\varepsilon}\hat{K}_A \theta_{\hat{A}} + O(\varepsilon^{3/2}) \tag{43}$$

$$\hat{\theta} = \hat{\theta}_0 + \varepsilon^{1/2}C_1 \hat{\theta}_1 + \varepsilon[C_1^2 \hat{\theta}_2 + C_1C_2 \hat{\theta}_3 + \hat{\theta}_{41} + Pr\, \hat{\theta}_{42}] + \varepsilon\left[K_A \hat{\theta}_A + \sum_\alpha K_\alpha \hat{\theta}_\alpha\right] + \frac{\hat{\varepsilon}}{\varepsilon}\hat{K}_A \hat{\theta}_{\hat{A}} + O(\varepsilon^{3/2}). \tag{44}$$

The location of the interface η_{IF} and $\hat{\eta}_{IF}$, respectively, is not known in advance. It is part of the solution and it is subject to the influence of non-zero order terms like all other quantities. Therefore a similar expansion holds for η_{IF}

$$\eta_{IF} = \eta_{IF0} + \varepsilon^{1/2}C_1\eta_{IF1} + \varepsilon[C_1^2\eta_{IF2} + C_1C_2\eta_{IF3} + \eta_{IF41} + Pr\,\eta_{IF42}] + \varepsilon\left[K_A\eta_{IFA} + \sum_\alpha K_\alpha\eta_{IF\alpha}\right] + \frac{\hat{\varepsilon}}{\varepsilon}\hat{K}_A\eta_{IF\hat{A}} + O(\varepsilon^{3/2}). \tag{45}$$

The relation between η_{IF} and $\hat{\eta}_{IF}$ is (C_2 according to Table 2)

$$\hat{\eta}_{IF} = \varepsilon^{1/2}C_2(\eta_{IF} - \eta_{IF0}) + O(\varepsilon^{3/2}). \tag{46}$$

2.3. *Zero- and first-order equations and their solutions*

Inserting equations (39) and (41)–(44) into the basic equations, equations (31) and (32), and equating terms of equal magnitude $C_i\varepsilon^j (i = 1, 2, \ldots;\ j = 0, 1/2, 1, \ldots)$ results in a hierarchy of sets of ordinary differential equations. The corresponding boundary conditions are easily derived from equations (18) and (19).

Special attention has to be given to the interfacial conditions, equations (33)–(38). A Taylor series expansion at the zero-order location is required to derive the higher order interfacial conditions. The first condition, equation (33), for example reads asymptotically (not all terms written down explicitly)

$$f_0'(\eta_0) + f_0''(\eta_0)\{\varepsilon^{1/2}C_1\eta_{IF1} + \varepsilon(C_1^2\eta_{IF2} + \cdots)\} + \tfrac{1}{2}f_0''(\eta_0)\{\varepsilon C_1^2\eta_{IF1}^2 + \cdots\} + \varepsilon^{1/2}C_1[f_1'(\eta_0) + f_1''(\eta_0)\{\varepsilon^{1/2}C_1\eta_{IF1} + \cdots\}] + \varepsilon[\ldots]$$
$$= \hat{f}_0'(\hat{\eta}_0) + \hat{f}_0''(\hat{\eta}_0)\{\varepsilon C_1C_2\eta_{IF1} + \cdots\} + \varepsilon^{1/2}C_1\hat{f}_1' + \varepsilon[\ldots].$$

The overall procedure results in the following equations, boundary conditions and interfacial conditions, respectively.

2.3.1. *Zero order; basic solution.*

$$f_0''' + 1 = 0 \tag{47a}$$

$$\hat{f}_0''' + 3\hat{f}_0\hat{f}_0'' - 2\hat{f}_0'^2 = 0 \tag{47b}$$

$$\theta_0'' = 0 \tag{48a}$$

$$\hat{\theta}_0'' + 3\hat{P}r\,\hat{f}_0\hat{\theta}_0' = 0 \tag{48b}$$

boundary conditions

$$\eta = 0: \quad f_0 = f_0' = \theta_0 - 1 = 0 \tag{49}$$

$$\hat{\eta} \to \infty: \quad \hat{f}_0' = \hat{\theta}_0 - 1 = 0; \tag{50}$$

interfacial conditions

$$f_0' = \hat{f}_0' \tag{51}$$

$$f_0'' = 0 \tag{52}$$

$$0 = \hat{f}_0 \tag{53}$$

$$\theta_0 = 0 \quad (54)$$

$$0 = \hat{\theta}_0 \quad (55)$$

$$\theta'_0 + 3Pr\,Hf_0 = 0. \quad (56)$$

2.3.2. $O(C_1\varepsilon^{1/2})$.

$$f_1''' = 0 \quad (57a)$$

$$\hat{f}_1''' + 3[\hat{f}_0\hat{f}_1'' + \hat{f}_1\hat{f}_0''] - 4\hat{f}'_0\hat{f}'_1 = 0 \quad (57b)$$

$$\theta_1'' = 0 \quad (58a)$$

$$\hat{\theta}_1'' + 3\hat{P}r[\hat{f}_0\hat{\theta}'_1 + \hat{f}_1\hat{\theta}'_0] = 0 \quad (58b)$$

boundary conditions

$$\eta = 0: \quad f_1 = f'_1 = \theta_1 = 0 \quad (59)$$

$$\hat{\eta} \to \infty: \quad \hat{f}'_1 = \hat{\theta}_1 = 0; \quad (60)$$

interfacial conditions (at $\eta = \eta_{\mathrm{IF0}}$)

$$f_0''\eta_{\mathrm{IF1}} + f'_1 = \hat{f}'_1 \quad (61)$$

$$f_0'''\eta_{\mathrm{IF1}} + f_1'' = \hat{f}_0'' \quad (62)$$

$$f_0 = \hat{f}_1 \quad (63)$$

$$\theta_1 = 0 \quad (64)$$

$$0 = \hat{\theta}_1 \quad (65)$$

$$\theta_0''\eta_{\mathrm{IF1}} + \theta'_1 + 3Pr\,H(f'_0\eta_{\mathrm{IF1}} + f_1) = 0. \quad (66)$$

It turns out that most of the non-zero order equations and their boundary and interfacial conditions are of a similar type which also holds for equations (57)–(66) above. In the Appendix all equations are listed by means of a general set of equations and the corresponding exceptions.

Altogether there are 11 sets of equations (four differential equations per set) with 11 boundary and interfacial conditions per set. The whole system has two parameters left (with only the first one being relevant as will be demonstrated afterwards)

$$Pr\,H$$

$$\hat{P}r.$$

It should be pointed out that the system treated in a non-asymptotic manner has six parameters even in the special case of constant properties, see, e.g. ref. [3].

The zero-order vapour equations can be solved analytically since interfacial conditions (51)–(56) let them be decoupled from the liquid phase. The solutions are

$$f_0 = -\tfrac{1}{6}\eta^3 + \tfrac{1}{2}\eta_{\mathrm{IF0}}\eta^2 \quad (67)$$

$$\theta_0 = 1 - \eta/\eta_{\mathrm{IF0}} \quad (68)$$

with the zero-order interface position given by

$$\eta_{\mathrm{IF0}} = (Pr\,H)^{-1/4}. \quad (69)$$

By means of equation (68) the zero-order heat transfer result can be given analytically as will be demonstrated in the next section.

The velocity and temperature distributions, equations (67) and (68), are those of the early theory of Nusselt [10] for water films which thus turn out to be the leading terms of an asymptotic theory.

All non-zero equations have to be solved numerically. Within each order ($\varepsilon^{1/2}C_1$, $\varepsilon C_1^2, \ldots$) the vapour and liquid equations are coupled by the unknown interface parameter $\eta_{\mathrm{IF}i}$. This unknown parameter as well as all initially unknown wall boundary conditions needed for a numerical integration (Runge–Kutta) of the equations are determined by a standard 'shooting method'.

Table 3. Numerical results. All quantities $\theta'_{i\mathrm{w}}$, $f''_{i\mathrm{w}}$ and $\eta_{\mathrm{IF}i}$ not included in this table are zero

$Pr\,H$	1	5	10
$\theta'_{0\mathrm{w}}$	−1.0	−1.495349	−1.778278
$\theta'_{42\mathrm{w}}$	−0.1	−0.029907	−0.017783
$\theta'_{\rho\lambda\mathrm{w}}$	0.5	0.747674	0.889138
$f''_{0\mathrm{w}}$	1.0	0.668740	0.562341
$f''_{1\mathrm{w}}$	−0.272407	−0.081624	−0.048676
$f''_{2\mathrm{w}}$	0.166731	0.022397	0.009476
$f''_{41\mathrm{w}}$	−0.091665	−0.012260	−0.005155
$f''_{42\mathrm{w}}$	−0.091667	−0.012260	−0.005155
$f''_{A\mathrm{w}}$	−0.375	−0.250778	−0.210878
$f''_{\rho\mu\mathrm{w}}$	−0.75	−0.501555	−0.421756
$f''_{\rho\lambda\mathrm{w}}$	0.16667	0.111457	0.093723
η_{IF0}	1.0	0.668740	0.562341
η_{IF1}	0.272407	0.081624	0.048676
η_{IF2}	−0.018319	−0.002471	−0.001051
η_{IF41}	0.075003	0.010032	0.004218
η_{IF42}	−0.091667	−0.012260	−0.005155
$\eta_{\mathrm{IF}A}$	0.125	0.083593	0.070293
$\eta_{\mathrm{IF}\rho\mu}$	0.25	0.167185	0.140585
$\eta_{\mathrm{IF}\rho\lambda}$	0.16667	0.111457	0.093723

Beforehand a careful inspection of all equations and boundary conditions reveals the following points.

(1) That a considerable number of equations have the trivial solution only. That they hold for all vapour equations of $O(\varepsilon C_1^2)$, $O(\varepsilon K_c)$ and $O(\varepsilon\hat{K}_A)$ and all vapour energy equations except $O(1)$, $O(\varepsilon\,Pr)$ and $O(\varepsilon K_{\rho\lambda})$.

(2) That the liquid Prandtl number $\hat{P}r$ does not affect the wall values of zero and first order, so that the whole problem is left as a one-parameter problem within the linear theory of this study.

This one and only parameter is $Pr\,H$. In Table 3 the numerical results are displayed for three different values of this parameter. Three quantities are given: $\theta'_{i\mathrm{w}}$, $f''_{i\mathrm{w}}$ and $\eta_{\mathrm{IF}i}$.

2.4. *Skin friction and heat transfer results*

The momentum transfer results in terms of the skin friction $c_f \equiv 2\tau^*_{\mathrm{w}}/\rho^*_{\mathrm{s}}U^{*2}$ are

$$c_f\varepsilon^{1/2} = 2\sqrt{2}\,Gr^{-1/4}x^{1/4}\Big[f_0'' + \varepsilon^{1/2}C_1f_1'' + \varepsilon(C_1^2f_2'' + C_1C_2f_3'' + f_{41}'' + Pr\,f_{42}'') + \varepsilon(K_Af_A'' + K_{\rho\mu}(f_0'' + f_{\rho\mu}'') + K_{\rho\lambda}f_{\rho\lambda}'') + \frac{\hat{\varepsilon}}{\varepsilon}\hat{K}_A f_{\hat{A}}''\Big]_{\mathrm{w}} + O(\varepsilon^{3/2}) \quad (70)$$

with Grashof number

$$Gr \equiv R^2 = \frac{g^* L^{*3}}{v_s^{*2}} \varepsilon(C-1). \quad (71)$$

The heat transfer results in terms of the Nusselt number

$$Nu = q_w^* L^* / \lambda_s^* (T_w^* - T_s^*)$$

are

$$Nu\,\varepsilon^{1/2} = -2^{-1/2} Gr^{1/4} x^{-1/4} [\theta_0' + \varepsilon^{1/2} C_1 \theta_1' + \varepsilon(C_1^2 \theta_2' + C_1 C_2 \theta_3' + \theta_{41}' + Pr\,\theta_{42}') + \varepsilon(K_A \theta_A' + K_{\rho\mu}\theta_{\rho\mu}' + K_{\rho\lambda}(\theta_0' + \theta_{\rho\lambda}')) + \frac{\hat{\varepsilon}}{\varepsilon} \hat{K}_A \theta_{\hat{A}}' + O(\varepsilon^{3/2}) \quad (72)$$

with the zero-order result

$$Nu\,\varepsilon^{1/2} = 2^{-1/2} Gr^{1/4} x^{-1/4} (Pr\,H)^{1/4}. \quad (73)$$

From equations (70) and (72) correction formulae to account for the influence of variable properties can be derived. With the constant property case characterized by $K_i \equiv O(i = \rho\mu,\ \rho\lambda,\ A)$ and $f_{\hat{A}w}'' = \theta_{Aw}' = \theta_{\rho\mu w}' = \theta_{\hat{A}w}' = 0$ the following ratios hold asymptotically:

$$\frac{c_f}{c_{fcp}} = 1 + \varepsilon\left[K_A \frac{f_{Aw}''}{f_{0w}''} + K_{\rho\mu}\left(1 + \frac{f_{\rho\mu w}''}{f_{0w}''}\right) + K_{\rho\lambda}\frac{f_{\rho\lambda w}''}{f_{0w}''}\right] + O(\varepsilon^{3/2}) \quad (74)$$

$$\frac{Nu}{Nu_{cp}} = 1 + \varepsilon K_{\rho\lambda}\left(1 + \frac{\theta_{\rho\lambda w}'}{\theta_{0w}'}\right) + O(\varepsilon^{3/2}). \quad (75)$$

It should be pointed out that the zero-order solution is the constant property case in the limit of vanishing heat transfer ($\varepsilon \to 0$). That is why ratios like c_f/c_{f0} (instead of c_f/c_{fcp}) would have terms of $O(\varepsilon^{1/2})$.

2.5. *Reference temperature and property ratio formulation*

In the heat transfer literature there are two empirical methods to account for variable property effects: the reference temperature and the property ratio method. The empirical constants in both methods can be deduced analytically by the asymptotic approach of this study.

2.5.1. *Reference temperature method.* In this method a temperature T_r (reference temperature) is specified at which the properties appearing in the dimensionless groups of the problem should be evaluated to get the variable property results by constant property formulae. Defining

$$T_r^* = T_s^* + j(T_w^* - T_s^*) \quad (76)$$

it is the factor j that has to be determined. The starting point is the constant property formula which for the heat transfer reads (see equation (72) and Table 3)

$$Nu_s = -(2\varepsilon)^{-1/2} x^{-1/4} Gr_s^{-1/4} [\theta_0'(Pr\,H_s) + \varepsilon\,Pr_s\,\theta_{42}'(Pr\,H_s)]_w = Nu_s(Gr_s, Pr\,H_s, Pr_s). \quad (77)$$

The index 's' means all properties in the dimensionless groups are evaluated at $T^* = T_s^*$. The same formula with all properties at $T^* = T_r^*$ describes the variable property Nusselt number Nu, so that the ratio Nu/Nu_{cp} reads

$$\frac{Nu}{Nu_{cp}} = \left[\frac{\lambda_r^*}{\lambda_s^*}\right]\left[\frac{Gr_r}{Gr_s}\right]^{1/4} \times \left[\frac{\theta_{0w}'(Pr\,H_r) + \varepsilon\,Pr_r\,\theta_{42w}'(Pr\,H_r)}{\theta_{0w}'(Pr\,H_s) + \varepsilon\,Pr_s\,\theta_{42w}'(Pr\,H_s)}\right] = 1 + \varepsilon j[\tfrac{1}{2}(K_\rho - K_\mu) + K_\lambda + F(Pr\,H_s)K_{Pr\,H}] \quad (78)$$

with

$$F(Pr\,H) = \left[\frac{Pr\,H}{\theta_{0w}'}\frac{d\theta_{0w}'}{d Pr\,H}\right]_s \quad (79)$$

and $K_{Pr\,H} = K_\mu - K_\lambda$, K_ρ, K_μ, K_λ according to equation (40).

Inserting θ_0 according to equations (68) and (69) into equation (79) one obtains $F(Pr\,H) = 1/4 =$ const. which leads to the final result

$$\frac{Nu}{Nu_{cp}} = 1 + \varepsilon j\left[\frac{1}{2}K_\rho - \frac{1}{4}{}_\mu K_\mu + \frac{3}{4}K_\lambda\right] = 1 + \varepsilon j\left[\frac{1}{2}K_{\rho\lambda} - \frac{1}{4}K_{Pr\,H}\right]. \quad (80)$$

Equating equations (80) and (75) one obtains

$$j = \frac{4K_{\rho\lambda}(1 + \theta_{\rho\lambda w}'/\theta_{0w}')}{2K_{\rho\lambda} - K_{Pr\,H}} = \frac{1}{1 - K_{Pr\,H}/(2K_{\rho\lambda})}. \quad (81)$$

In equation (81) the analytical solution for $\theta_{\rho\lambda}$ (see equation (A20)) was used, which reads

$$\theta_{\rho\lambda} = -(Pr\,H)^{1/2}\eta^2/2 + (Pr\,H)^{1/4}\eta/2$$

and results in $\theta_{\rho\lambda w}'/\theta_{0w}' = -1/2$. Assuming $K_{Pr\,H} \equiv K_\mu - K_\lambda$ to be small, equation (81) can be expanded

$$j = 1 + \frac{K_{Pr\,H}}{2K_{\rho\lambda}} + O(K_{Pr\,H}^2). \quad (82)$$

This surprising result means for fluids with $K_\mu = K_\lambda$ ($K_{Pr\,H} = 0$) the reference temperature is the wall temperature ($j = 1$, see equation (76)) and for fluids with $K_\mu \approx K_\lambda$ it is close to the wall temperature.

Data for film boiling of water at atmospheric pressure for example are: $K_\mu = 1.25$, $K_\lambda = 1.38$ resulting in $T_r^* = T_s^* + 0.83(T_w^* - T_s^*)$.

It should be pointed out that this reference temperature only holds for the heat transfer results. One has to deduce a different reference temperature if the constant property c_f result is used in the variable property case. This inherent feature of the reference temperature method is clearly revealed by the asymptotic method.

2.5.2. *Property ratio method.* In this method the constant property results are multiplied by a power of some pertinent property (or properties) evaluated at two different temperatures. The unknown power (or powers) was (were) determined empirically in the past. To the author's knowledge this method has not been applied to film boiling so far.

By means of the asymptotic theory the choice of properties is obvious and the unknown power can be deduced analytically.

The property ratio formula must be of the general form

$$Nu = Nu_{cp}\left[\frac{\rho_w^* \lambda_w^*}{\rho_s^* \lambda_s^*}\right]^{n_{\rho\lambda}} \quad (83)$$

which asymptotically is

$$\frac{Nu}{Nu_{cp}} = 1 + \varepsilon K_{\rho\lambda} n_{\rho\lambda} + O(\varepsilon^2). \quad (84)$$

Comparing this equation with the variable property result equation (75) gives for $n_{\rho\lambda}$

$$n_{\rho\lambda} = 1 + \frac{\theta'_{\rho\lambda w}}{\theta'_{0w}} = \frac{1}{2} \quad (85)$$

resulting in the simple property ratio formula

$$\frac{Nu}{Nu_{cp}} = \sqrt{(\rho\lambda)_w}. \quad (86)$$

3. DISCUSSION

There are four important features of the asymptotic approach to complex two-face flows that should be emphasized.

(1) The typical advantage of a perturbation technique holds: the results are general in the sense that a specification to certain flow cases is made in the results only (by specifying ε, $\hat{\varepsilon}$ and the fluid through K_ρ, $K_{\hat{\rho}}$, $K_\eta, \ldots$).

(2) The influence of the physical properties can be checked separately and a hierarchy of property influences is established in terms of first-order and higher order effects.

(3) The numerical solutions are more general and easier to obtain compared to a non-asymptotic approach. Only one solution parameter is left up to first order. There is no need for iterative solution procedures and the unknown interface position is given explicitly.

(4) All information is extracted from the basic equations. Based on these results well-known empirical methods to account for variable property effects can be understood as theoretical methods (Section 2.5).

It should be pointed out that aside from the specific

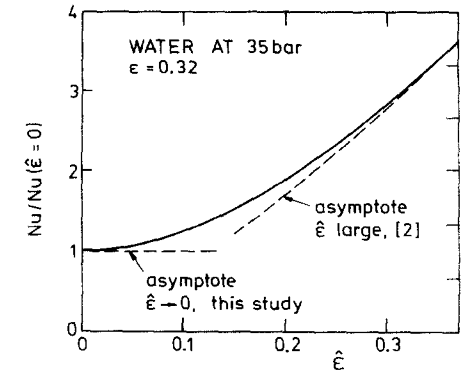

FIG. 2. Subcooling influence, results and asymptote for large $\hat{\varepsilon}$ from ref. [2].

numerical numbers provided by this theory a hierarchy of influences and effects is established. One example is the following.

Sparrow and Cess [2] investigated the influence of subcooling on film boiling heat transfer. They showed that for strong subcooling the heat transfer is asymptotically equal to that for pure free convection (Fig. 2). The asymptote for vanishing subcooling is provided by equation (72) of this study

$$Nu(\hat{\varepsilon}) = Nu(0) + \frac{dNu}{d\hat{\varepsilon}}\hat{\varepsilon} + O(\hat{\varepsilon}^2)$$

$$= Nu(0) + \frac{\hat{\varepsilon}}{\varepsilon}\hat{K}_A \theta'_{\hat{A}} + O(\hat{\varepsilon}^2). \quad (87)$$

With $\theta'_{\hat{A}} = 0$ the asymptote for $\hat{\varepsilon} \to 0$ in Fig. 2 follows. Asymptotically that means that subcooling is a higher order effect.

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APPENDIX

A general set of equations that holds for all non-zero order functions (exceptions listed below) reads ($i = 2, 3, 41, 42, A, \rho\mu, \rho\lambda, \hat{A}$)

$$f_i''' = 0 \qquad \text{(A1a)}$$

$$\hat{f}_i''' + 3[\hat{f}_0\hat{f}_i'' + \hat{f}_i\hat{f}_0''] - 4\hat{f}_0'\hat{f}_i' = 0 \qquad \text{(A1b)}$$

$$\theta_i'' = 0 \qquad \text{(A2a)}$$

$$\hat{\theta}_i'' + 3\hat{Pr}[\hat{f}_0\hat{\theta}_i' + \hat{f}_i\hat{\theta}_0'] = 0 \qquad \text{(A2b)}$$

boundary conditions

$$\eta = 0: \quad f_i = f_i' = \theta_i = 0 \qquad \text{(A3)}$$

$$\hat{\eta} \to \infty: \quad \hat{f}_i' = \hat{\theta}_i = 0; \qquad \text{(A4)}$$

interfacial conditions

$$f_0''\eta_{\mathrm{IF}i} + f_i' = \hat{f}_i' \qquad \text{(A5)}$$

$$f_0'''\eta_{\mathrm{IF}i} + f_i'' = 0 \qquad \text{(A6)}$$

$$0 = \hat{f}_i \qquad \text{(A7)}$$

$$\theta_i = 0 \qquad \text{(A8)}$$

$$0 = \hat{\theta}_i \qquad \text{(A9)}$$

$$\theta_0''\eta_{\mathrm{IF}i} + \theta_i' + 3Pr\,H(f_0'\eta_{\mathrm{IF}i} + f_i) = 0. \qquad \text{(A10)}$$

Exceptions (each line starts with the equation to be replaced).

$\underline{O(C_1^2\varepsilon)}$

$$\text{(A5)} \to f_0''\eta_{\mathrm{IF2}} + \tfrac{1}{2}f_0'''\eta_{\mathrm{IF1}}^2 + f_1''\eta_{\mathrm{IF1}} + f_2' = \hat{f}_2' \qquad \text{(A11)}$$

$$\text{(A6)} \to f_0'''\eta_{\mathrm{IF2}} + \tfrac{1}{2}f_0''''\eta_{\mathrm{IF1}}^2 + f_1'''\eta_{\mathrm{IF1}} + f_2'' = \hat{f}_1'' \qquad \text{(A12)}$$

$$\text{(A7)} \to f_0'\eta_{\mathrm{IF1}} + f_1 = \hat{f}_2 \qquad \text{(A13)}$$

$$\text{(A10)} \to \theta_0''\eta_{\mathrm{IF2}} + \tfrac{1}{2}\theta_0'''\eta_{\mathrm{IF1}}^2 + \theta_1''\eta_{\mathrm{IF1}} + \theta_2'$$
$$+ 3Pr\,H(f_0'\eta_{\mathrm{IF2}} + \tfrac{1}{2}f_0''\eta_{\mathrm{IF1}}^2 + f_1'\eta_{\mathrm{IF1}} + f_2) = 0. \qquad \text{(A14)}$$

$\underline{O(C_1C_2\varepsilon)}$

$$\text{(A7)} \to 0 = \hat{f}_0'\eta_{\mathrm{IF1}} + \hat{f}_3. \qquad \text{(A15)}$$

$\underline{O(\varepsilon)}$

$$\text{(A1a)} \to f_4''' + 3f_0f_0'' - 2f_0'^2 = 0 \qquad \text{(A16)}$$

$$\text{(A2a)} \to \theta_{42}'' + 3f_0\theta_0' = 0. \qquad \text{(A17)}$$

$\underline{O(K_A\varepsilon)}$

$$\text{(A1a)} \to f_A''' - \theta_0 = 0. \qquad \text{(A18)}$$

$\underline{O(K_{\rho\mu}\varepsilon)}$

$$\text{(A1a)} \to f_{\rho\mu}''' + (f_0''\theta_0)' = 0. \qquad \text{(A19)}$$

$\underline{O(K_{\rho\lambda}\varepsilon)}$

$$\text{(A2a)} \to \theta_{\rho\lambda}'' + (\theta_0\theta_0')' = 0. \qquad \text{(A20)}$$

$\underline{O\left(\frac{\hat{\varepsilon}}{\varepsilon}\hat{K}_A\right)}$

$$\text{(A1b)} \to \hat{f}_A''' + 3[\hat{f}_0\hat{f}_A'' + \hat{f}_0''\hat{f}_A]$$
$$- 4\hat{f}_0'\hat{f}_A' + (1 - \hat{\theta}_0) = 0. \qquad \text{(A21)}$$

ANALYSE ASYMPTOTIQUE DU FILM LAMINAIRE D'EBULLITION SUR DES PLANS VERTICAUX EN INCLUANT LES EFFETS DE PROPRIETE VARIABLE

Résumé—L'ébullition avec film laminaire est étudiée pour démontrer les avantages de l'approche asymptotique pour des problèmes complexes d'écoulement diphasique avec transfert de chaleur. En introduisant deux paramètres de perturbation respectivement pour la surchauffe et le sous-refroidissement, on obtient une solution régulière de perturbation avec seulement deux paramètres. Le effets des propriétés variables sont inclus asymptotiquement. Les solutions non asymptotiques sont bien moins générales car, même pour des propriétés constantes, c'est un problème à six paramètres qui ne donne que des solutions spécifiques.

EINE ASYMPTOTISCHE BETRACHTUNG DES LAMINAREN FILMSIEDENS AN SENKRECHTEN PLATTEN UNTER BERÜCKSICHTIGUNG DES EINFLUSSES VARIABER STOFFWERTE

Zusammenfassung—Am Beispiel des laminaren Filmsiedens sollen die Vorteile der asymptotischen Theorie bei der Behandlung vergleichsweise komplexer Zweiphasen-Strömungen und Wärmeübertragungsprobleme demonstriert werden. Nach Einführung von zwei Störparametern zur Erfassung der Dampfüberhitzung bzw. Flussigkeitsunterkühlung kann eine reguläre Störungsrechnung durchgeführt werden, die zu einer Lösung mit nur zwei Parametern führt. Die Effekte variabler Stoffwerte sind dabei in einem asymptotischen Sinne berücksichtigt. Nichtasymptotische Lösungen sind sehr viel weniger allgemeingültig, da schon im Falle konstanter Stoffwerte sechs Lösungsparameter auftreten, so daß nur spezielle Lösungen bestimmt werden können.

АСИМПТОТИЧЕСКИЙ АНАЛИЗ ЛАМИНАРНОГО ПЛЕНОЧНОГО КИПЕНИЯ НА ВЕРТИКАЛЬНЫХ ПЛАСТИНАХ С УЧЕТОМ ПЕРЕМЕННОСТИ СВОЙСТВ

Аннотация—На примере изучения ламинарного пленочного кипения показаны преимущества асимптотического метода расчета сложных задач двухфазного течения и теплообмена. Используя два параметра возмущения для описания перегрева и недогрева, методом возмущений получено решение, содержащее только два параметра. Асимптотически учитываются эффекты переменности свойств. Неасимптотические решения являются гораздо менее общими, так как даже в случае постоянных свойств для получения частных решений необходимо учитывать шесть параметров.